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THE CRITICAL PRESSURE FOR FLOW IN A POROUS MEDIUM†

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Bingham flow in a porous medium is considered. This can be modelled by a random structure whose dimensions are large compared with the local scale. The principal term of the asymptotic form of the critical pressure at which the liquid starts to move in this limit is computed explicitly. © 1998 Elsevier Science Ltd. All rights reserved.

For Bingham flow through a porous medium, the relation between the flow rate and the pressure, which is equal to the difference of the volume density of mass forces and the pressure gradient, is non-linear and contains a critical parameter: seepage only takes place at pressures above a certain threshold value. Certain mixtures of liquids which do not initially possess rigid-plastic properties behave similarly. For example, water in which there are air bubbles might not flow through a porous medium because the pores are sealed, and a pressure which can push the bubbles through the pores, overcoming the surface tension, is required before flow can begin.

The Darcy relation between the pressure and the flow rate, which in general is non-linear, has been established experimentally for practical porous media. One can also use a purely theoretical approach, investigating the fluid flow in a porous region comprising a system of numerous pores and averaging, that is, determining the principal term of the asymptotic form of the solutions as the number of pores increases. It is sometimes possible to use this method to obtain acceptable estimates for the parameters in the Darcy relation and their connection with the properties of the liquid and the structure of the medium. No less important a result of such an investigation, albeit only qualitative, is the proof that it is possible to transfer from the problem at the microlevel to the simpler equations of the theory of seepage through a porous medium. This is the method used to prove Darcy's law for linearly viscous liquids in [1–3], where the pore structure was taken as a small-scale region with a periodic structure. These results have also been extended to random structures [4, 5]. Bingham flow has been averaged [6] only over periodic structures. There are specific features of Bingham liquids which make it difficult to average over random porous regions.

In this paper we examine a problem which illustrates one of these difficulties and which shows what new effects one can expect compared with the periodic case when averaging is used.

1. STATEMENT OF THE PROBLEM

Consider a porous medium consisting of a large number of random pores aligned in the same direction. A section of this system of pores by a plane perpendicular to their direction is a random two-dimensional region Ω, consisting of separate elements of random shape, size and position, but which might also be connected. Fibrous porous materials have structures of this kind. They are also appropriate to use in a general situation where the liquid crossflow in transverse pores is insignificant. We will model the region Ω with a random Bernoulli chequerboard, constructed as follows: a square of size $r \times r$ is divided into a large number of cells of size $a \times a$, $a \ll r$, and then independently of the other cells, each cell is left where it is with probability p or removed, with probability $1 - p$; the region $\Omega = \Omega^{(r)}$ is the set of cells that remain. To fix our ideas, we shall take r to be a large parameter, and the size of the cell a as a fixed number. While this is not the best geometric model of real porous media we could use, it is often employed in investigations because it is easy to visualize and simple to describe.

Suppose that the liquid fills the entire pore space and is acted upon in a longitudinal direction by a constant pressure. In this case, the steady-motion problem reduces to a two-dimensional problem in Ω for the longitudinal velocity component of the liquid with a Dirichlet condition on the pore boundaries. The critical pressure (CP) at which the liquid starts to move is a random number which depends on

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the parameter r. It will be shown that the principal term of the asymptotic form of the CP as $r \to \infty$ has the form $C/\sqrt{(\ln r)}$ with non-random constant C which can be computed explicitly.

If the pore structure is periodic, the CP would have a positive limit as $r \to \infty$. Allowing for the randomness of the structure might therefore have a decisive influence on the averaged seepage properties and result in degeneracy of the CP.

This degeneracy is easy to explain in qualitative terms. As the pressure increases, the Bingham liquid first begins to move in the widest pores. Thus the value of the CP does not depend on the typical portions of pore structure, but on the presence of abnormally large pores. The more representative a portion of the porous medium is, the sooner a particularly large pore will be found and therefore the lower the pressure at which the liquid starts to move. Thus the CP degenerates. The order of degeneracy and, especially, the explicit asymptotic form, depend on the specific model of the random structure. The decisive feature here is not the average statistical properties of the structure, such as the typical pore size, but the probability of large deviations. For this reason it might be ineffective to use averaging methods for Bingham flow in a real porous medium, since it is never possible to obtain the statistics of infrequent deviations reliably from the results of direct observation.

2. BASIC DEFINITIONS AND FORMULATION OF THE RESULT

The liquid velocity in the cylindrical region described above has just one component, u, which depends on the coordinates (x, y) on Ω. The strain rate tensor has just two important components, which can be expressed in terms of the components of the gradient of the function u. We define the rheology of the Bingham liquid, specifying its dissipative potential by the formula

$$W(\nabla u) = k|\nabla u| + \mu|\nabla u|^2/2, \quad \nabla \equiv (\partial/\partial x, \partial/\partial y)$$

The two non-zero components of the tensor of tangential stresses are defined in terms of W by the equation $\tau = \partial W/\partial \nabla u$ whenever the function W is differentiable; at the conical point $\nabla u = 0$, however, they can take values $|\tau| \leqslant k$. The specific form of the second term in the dissipative potential will be unimportant below, provided that W increases fast enough and is convex.

The problem of the steady motion of a viscous liquid in a cylinder has a variational formulation [7, 8]: the longitudinal component of the velocity gives a minimum of the functional

$$J(u) = \int_\Omega [W(\nabla u) - iu]dxdy$$

where i is the pressure, and the lower bound of J is found from the functions $u(x, y)$, which are equal to zero on the boundary $\partial\Omega$ of the cylinder section.

Owing to the conical singularity of the dissipative potential, the minimum of J is zero not only at $i = 0$, but also for sufficiently small $i > 0$. As the pressure increases, the minimum becomes negative, and the minimizant $u(x, y)$ becomes non-zero. The formal definition of the CP at which this happens has the form

$$i^*(\Omega) = \sup\{i: \inf J = 0\} = \inf\{i: \inf J < 0\}$$

Apart from J we consider the functional

$$J_0(u) = \int_\Omega [k|\nabla u| - iu]dxdy$$

The solution of the problem is a continuous function of i, and so for near-critical pressures the second term in the expression for W is small. Hence, we can use J_0 instead of the functional J in the definition of i^*. This can be stated exactly as follows: for $i < i^*$ $\inf J_0 = 0$, and for $i > i^*$ $\inf J_0 = -\infty$. Hence we have the following variational definition of the CP

$$i^*(\Omega) = \inf\left\{k\int_\Omega |\nabla u|\,dxdy: \int_\Omega u dxdy = 1\right\} \tag{2.1}$$

We shall seek the lower bound here among functions which satisfy the normalization condition and are equal to zero on $\partial\Omega$. Note that the functional J_0 does not grow fast enough, and for $i \geqslant i^*$ its lower bound cannot be achieved. All the more so, its minimizant like that in (2.1), is not necessarily identical with the solution of the original problem.

Formula (2.1) can be used to compute $i^*(\Omega)$ explicitly only if the region Ω is sufficiently simple, like a circle. There has been a great deal of research on the flow of viscous liquids, including Bingham liquids, in circular pipes [7, 8]. It is impossible to obtain an analytic solution for the random chequerboard structure $\Omega^{(r)}$ described above, although the principal term of the asymptotic form $i^*(\Omega)^{(r)}$ as $r \to \infty$ can be found explicitly and has the form

$$i^*(\Omega^{(r)}) \approx k\lambda_d V_r^{-1/d}, \quad V_r = da^d \ln(r/a)(\ln(1/p))^{-1} \tag{2.2}$$

$$\lambda_d = d\sqrt{\pi}(\Gamma(1 + d/2))^{-1/d}$$

This is the main result of this paper. Here $d = 2$ denotes the dimension of problem (2.1), which can be formulated formally in a space of any dimension, and $\lambda_2 = 2\sqrt{\pi}$. The symbol "$\approx$" in (2.2) is used to denote convergence in probability terms. Its exact meaning is that for any prescribed number $\varepsilon > 0$, the ratio of the left and right-hand sides of relation (2.2) differs from unity by less than ε with a probability which tends to unity as $r \to \infty$. Relation (2.2) will be proved below under the additional condition $p \ll 1$, that is, for low porosites, although it also holds for finite p.

The mathematical problem (2.1) can be given a different physical interpretation. Suppose that the lower half $z < 0$ of a three-dimensional space with coordinates (x, y, z) is filled with water, and the upper half with a porous material which is not wetted by water and in which the pores contain air. Suppose that all the voids of this material are pointing vertically and occupy the region $\Omega \times [0, +\infty)$, where Ω is the set of holes in the (x, y) plane. Surface tension prevents the water from penetrating through the holes of Ω into the upper half-space. When the pressure in the water and the air is identical, the interface between them lies in the plane $z = 0$. But if the pressure in the water is greater, the interface moves up and is a surface $z = Z(x, y)$, the edges of which lie on the edges of the holes. If the pressure drop is significant, the forces of surface tension are insufficient to keep the system in equilibrium, and water starts to enter the pore material. The shape of the equilibrium interface $Z(x, y)$ is found by minimizing the functional

$$I(Z) = \int_\Omega [\gamma(1+|\nabla Z|^2)^{1/2} - PZ]dxdy$$

where γ is the surface tension coefficient, and P is the constant pressure difference between the water and the air.

The functional I is the sum of the surface and potential energies of dome-shaped drops which have penetrated into the upper half-space under pressure. When minimizing the energy, the functions $Z(x, y)$, which are zero on the boundary of Ω, are taken as test functions. The term with surface energy in I increases at the same rate as the linear term. Thus at pressures P above a certain critical value P^*, the minimum of I goes to minus infinity, and the equilibrium form of the phase interface ceases to exist.

It follows from the inequalities $|\nabla Z| \leqslant (1 + |\nabla Z|^2)^{1/2} \leqslant 1 + |\nabla Z|$ that the lower limit of I tends to $-\infty$ with the lower limit of the functional

$$I_0(Z) = \int_\Omega [\gamma|\nabla Z| - PZ]dxdy.$$

It is the same, in different notation, as the functional J_0 introduced above in connection with a Bingham liquid. Thus the critical pressure drop P^* is given by the same formula (2.1) as the critical pressure i^*, with the parameter k replaced by the surface tension coefficient γ.

The asymptotic form of the least eigenvalue of the Laplace operator $(-\Delta)$ in a random perforated region was investigated previously [9, 10]. For this value we can use Rayleigh's variational formula, which is similar in form to (2.1) but contains second powers of ∇u and u in the functional and the normalizing condition. This means that we can use the methods devised in [9, 10] and earlier publications to derive the asymptotic formula (2.2).

It should be noted that the model used in [9] differed from the random chequerboard model we have used. The region Ω was a d-dimensional cube with edge r from which randomly positioned spheres of constant radius were removed. The centres of the spheres had a Poisson distribution with density $\nu > 0$. The asymptotic form of the CP i^* as $r \to \infty$ for this model can also be calculated. To compare it with (2.2), we give the final result

$$i^* = k\lambda_d(d(\ln r)/\nu)^{-1/d}$$

where the constant λ_d has the same value as for the chequerboard model.

3. AN UPPER LIMIT OF THE CRITICAL PRESSURE

We will prove that the probability of the event

$$i^*(\Omega^{(r)}) > k\lambda_d V_r^{-1/d}(1+\varepsilon) \tag{3.1}$$

tends to zero as $r \to \infty$ for any $\varepsilon > 0$. This, together with the corresponding lower probability limit, gives the asymptotic equality (2.2).

Let Ω, Ω' be two regions in the plane $\mathbb{R}^d$, and $\Omega' \subset \Omega$. Then it follows at once from the definition of the CP that $i^*(\Omega) \leqslant i^*(\Omega')$. We will use this inequality to construct an upper limit $i^*(\Omega^{(r)})$ selecting the region Ω' inscribed in $\Omega^{(r)}$ in such a way that the quantity $i^*(\Omega')$ can be computed explicitly.

We will need a formula for the CP in a circular pipe. This has the form

$$i^* = k\lambda_d S^{-1/d}$$

where S is the cross-section area of the pipe.

A full derivation of this formula is given, for example, in [8]. At a physical level of rigor, it can be derived by equating the total pressure iS to the friction force of the liquid at the wall per unit length of the pipe. When $i = i^*$ this force is equal to the product of the yield limit k and the circumference of the cross-section, or $2\sqrt{(\pi S)}$ when $d = 2$. If $d = 2$, the surface area of a d-dimensional sphere must be used instead of the circumference. For a sphere of unit volume, the surface area can be expressed in terms of the Γ-function and is equal to λ_d.

We will cover the square $r \times r$ which contains $\Omega^{(r)}$, with identical non-interesting circles, each of area S. We will choose the quantity $S = S(r)$ later, but first assume that $a^d \ll S \ll r^d$. The covering is so made that none of the $a \times a$ cells into which the square is divided intersects two circles simultaneously, and also that as $r \to \infty$ the number of circles N satisfies the inequality $N \geqslant Cr^d/S$ for some constant C. If at least one circle lies entirely inside the random set $\Omega^{(r)}$, it can be taken as the auxiliary region Ω'. Then we have the inequality

$$i^*(\Omega^{(r)}) \leqslant k\lambda_d S^{-1/d} \tag{3.2}$$

The number of cells n which have a non-empty intersection with one of the circles can change from circle to circle, but for sufficiently large S we have the uniform estimate

$$n \leqslant (1+\delta)S/a^d, \quad \delta > 0$$

The probability that a fixed circle is contained in $\Omega^{(r)}$, that is, that none of the cells which intersect it is discarded, is equal to p^n. The probability of the opposite happening is equal to $1-p^n$, and the probability that none of the N circles lies entirely in $\Omega^{(r)}$ is equal to the product of N such expressions, since the configurations for different circles are independent. This product does not exceed the quantity

$$(1-p^{(1+\delta)S/a^d})^{Cr^d/S} \tag{3.3}$$

and this therefore gives an upper limit for the probability of the opposite of (3.2). We now put $S = V_r(1+\delta)^{-2}$. Clearly, with this choice of the function $S(r)$ the quantity (3.3) will tend to zero. To complete the proof, it only remains to define $\delta = \delta(\varepsilon)$ so that the event opposite to (3.2) is the same as (3.1).

4. A LOWER LIMIT OF THE CRITICAL PRESSURE

Finding a lower limit for the CP is a very much more difficult problem. As in [9], we will first consider the low-porosity case, and then give an outline of the proof for finite p. The construction of a lower limit is based on an isoperimetric inequality, which we will give in the form of the following lemma ([8], Lemma 2.4).

Lemma 1. The critical pressure $i^*(\Omega)$ in a pipe of arbitrary cross-section Ω is no less than the critical pressure for a circular pipe of the same cross-section area

$$i^*(\Omega) \geqslant k\lambda_d |\Omega|^{-1/d}$$

where $|\Omega|$ is the area of Ω.

The analogous statement for the least eigenvalue of the Laplace operator in the region Ω with a Dirichlet condition on the boundary is well known, and was proved by Rayleigh (for a modern version of this proof see [11]). The same proof, with only slight changes, can be used for Lemma 1, and so it will not be given here.

In the low-porosity case, the basic relation (2.2) follows from the fact that the probability of the event

$$i^*(\Omega^{(r)}) \leqslant k\lambda_d V_r^{-1/d}(1-\varepsilon) \tag{4.1}$$

tends to zero for any $\varepsilon > 0$ if $r \to \infty, p \to 0$.

To prove the last statement, let $A^{(r)}$ denote the finite set consisting of the centres of the cells into which the $r \times r$ square was divided. We will number the cells using the elements $A^{(r)}$. For each $\alpha \in A^{(r)}$ we will denote by Ω_α the connected component of the random set $\Omega^{(r)}$ in which the corresponding cell lies ($\Omega_\alpha \neq \varnothing$ if the cell was discarded and is not in $\Omega^{(r)}$). In percolation theory [12] the regions Ω_α are called clusters. The CP for the entire structure is obviously equal to the lowest of the critical pressures in individual cylinders with non-empty sections Ω_α. By Lemma 1 we have the estimate

$$i^*(\Omega^{(r)}) = \min\{i^*(\Omega_\alpha),\ \alpha \in A^{(r)}\} \leqslant k\lambda_d(\max\{|\Omega_\alpha|,\ \alpha \in A^{(r)}\})^{-1/d} \tag{4.2}$$

Thus, we can obtain an upper-limit for the CP in terms of the area of the largest of the clusters Ω_α, $\alpha \in A^{(r)}$. Percolation theory gives a probability estimate of the size of a finite cluster ([12]): for any $S > 0$ the probability that the cluster Ω_α contains more than S/a^d cells is not greater than $(Cp)^{S/ad}$ for some constant C which depends only on the dimension of the problem. This estimate is for an unbounded random lattice, but is even more valid if the structure is confined to a square $r \times r$. It only has a meaning for sufficiently small values of $p < C^{-1}$.

The event $\{|\Omega_\alpha|, \alpha \in A^{(r)}\} > S$ in probability space is the sum of events $|\Omega_\alpha| > S$ over all $\alpha \in A^{(r)}$. Its probability is therefore no greater than the sums of their probabilities can be estimated by the quantity

$$(r/a)^d (Cp)^{S/a^d} = \exp\{(S/a^d)\ln C - (S/a^d)\ln(1/p) + d\ln(r/a)\} \tag{4.3}$$

We will put $S = V_r(1+\delta)$ with some $\delta > 0$. By inequality (4.2), the quantity (4.3) is an upper limit for the probability of the event (4.1) with the required choice of $\delta = \delta(\varepsilon)$. The quantity (4.3) tends to zero as $r \to \infty$ under the condition $\delta \ln(1/p) > (1+\delta)\ln C$. For any values of δ this condition is satisfied only if $p \to 0$. Thus the required lower probability limit has been obtained in the low porosity case.

For finite p we can use the renormalization method developed in [9], which reduces the problem to the case of small p to prove relation (2.2). A complete account of the method applied to the problem of the principal eigenvalue of a Laplace operator is given in [9]. No fundamental changes are needed to calculate the CP.

We will therefore just give an outline of the argument and omit the details.

It can be verified that the principal term of the asymptotic form of expression (2.2) as $r \to 0$ does not change under the transformation $a \to \lambda a, p \to p^{\lambda d}$, where $\lambda > 0$ and $\ln\lambda \ll \ln(r/a)$. The renormalization involves replacing the region $\Omega^{(r)}$ by a random structure of the same type, but consisting of larger cells and with lower porosity. Then in the limit the new structure will possess the same CP.

In the renormalization we introduce an intermediate scale $r_1, a \ll r_1 \ll r$ and divide the original square into square blocks of size $r_1 \times r_1$. For a fixed configuration of remaining and discarded cells $\Omega^{(r)}$, each block can be classified as "empty", "full" or "intermediate", independently of the others. Blocks which do not contain one discarded cell are empty. The probability p_1 of such an event is computed from the formula $\ln(1/p_1) = (r_1/a)^d \ln(1/p)$ and tends to zero. Full blocks are those with a small proportion of remaining cells. They can be removed from the structure without substantially affecting the CP. The strict derivation of this statement from variational principle (2.1) is based on the Poincaré inequality, while it can be explained in physical terms by noting that in these blocks the liquid lies in relatively narrow pores and the pressures required to move it are much greater than the critical pressure.

If there were no intermediate blocks, once full blocks had been removed, the renormalized structure would be of the same kind as the original one but would possess porosity p_1 instead of p and cells of size $r_1 \times r_1$ instead of $a \times a$. The results proved for low porosity would still apply and, if $\ln r_1 \ll \ln r$, formula (2.2) would be proved.

We can eliminate intermediate blocks by introducing a hierarchy of sizes $a \ll r_1 \ll \ldots \ll r_m \ll r$, where $m \gg 1$. This construction can be interpreted in the following way. Intermediate blocks cannot be eliminated at the first stage of renormalization because they do not "retain" the liquid much better than empty blocks of the same size. However, their retaining capacity is much greater than that of empty blocks of the next level. Thus, blocks of size $r_2 \times r_2$, which consist of non-empty blocks of the previous size, are "full" blocks and are removed from the structure. As a result, at high levels $m \gg 1$ there remains a random region of low porosity for which the CP is not very much more than the original CP and has the asymptotic form (2.2).

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